



The Vector Linear Fractional Initialization Problem

Tom T. Hartley
The University of Akron, Akron, Ohio

Carl F. Lorenzo
Glenn Research Center, Cleveland, Ohio

National Aeronautics and
Space Administration

Glenn Research Center

Available from

NASA Center for Aerospace Information
7121 Standard Drive
Hanover, MD 21076
Price Code: A03

National Technical Information Service
5285 Port Royal Road
Springfield, VA 22100
Price Code: A03

The Vector Linear Fractional Initialization Problem

Tom T. Hartley
Department of Electrical Engineering
The University of Akron
Akron, OH 44325-3904

Carl F. Lorenzo
National Aeronautics and Space Administration
Glenn Research Center
Cleveland, OH 44135

Abstract

This paper presents a solution to the initialization problem for a system of linear fractional-order differential equations. The scalar problem is considered first, and solutions are obtained both generally and for a specific initialization. Next the vector fractional order differential equation is considered. In this case, the solution is obtained in the form of matrix F-functions. Some control implications of the vector case are discussed. The suggested method of problem solution is shown via an example.

Introduction

Fractional order systems, or systems containing fractional derivatives and integrals, have been studied by many in the engineering area (Heaviside, 1922; Bush, 1929; Goldman, 1949; Holbrook, 1966; Starkey, 1954; Carslaw and Jeager, 1948; Scott, 1955; and Mikusinski, 1959). Additionally, very readable discussions, devoted specifically to the subject, are presented by Oldham and Spanier (1974) and Miller and Ross (1993). It should be noted that there are a growing number of physical systems whose behavior can be compactly described using fractional system theory. Of specific interest to electrical engineers are long electrical lines (Heaviside, 1922), electrochemical processes (Ichise, Nagayanagi, and Kojima, 1971; Sun, Onaral, and Tsao, 1984), dielectric polarization (Sun, Abdelwahab, and Onaral, 1984), colored noise (Mandelbrot, 1967), viscoelastic materials (Bagley and Calico, 1991; Koeller, 1984; Koeller, 1986; Skaar, Michel, and Miller, 1988), and chaos (Hartley, Lorenzo, and Qammar, 1995).

With the growing number of applications, it is important at this time to establish a clear system theory for these fractional order systems, so that they may be accessible to the general engineering community. This topic is addressed in the present paper with regard to the initialization of vector systems, while the generalized impulse response was discussed in (Hartley and Lorenzo, 1998b). The first section presents the scalar fractional order differential equation, where both initial condition and forced responses are obtained. Next the vector fractional-order differential equation is considered. A simple example demonstrating the use of the vector system theory is given. The paper concludes with a discussion of some system theoretic implications of the vector realization.

The Scalar Initialization Problem

The general linear scalar constant-coefficient fractional differential equation, for $0 < q < 1$, is given as

$${}_a d_t^q x(t) = Ax(t) + Bu(t), \quad (1)$$

where it is assumed that the system is quiescent at time a , that is $x(t)=0$ for all $t < a$. The left side of Equation (1) should be interpreted to be the q -th derivative of $x(t)$, starting at time a and continuing until time t , namely

$${}_a d_t^q x(t) \equiv \frac{d}{dt} \left[\frac{1}{\Gamma(1-q)} \int_a^t \frac{x(\tau)}{(t-\tau)^q} d\tau \right].$$

Now, the initialization problem (Lorenzo and Hartley, 1998), starts at some time c , with $a \leq c < t$. Choosing $c = 0$, for ease of using Laplace transforms, Equation (1) can also be written as

$${}_0 d_t^q x(t) + {}_a d_0^q x(t) = Ax(t) + Bu(t). \quad (2)$$

where the fractional derivative has been broken into positive and negative time parts, respectively. It is important to point out that the fractional derivative is generally not a local operator, as is the integer derivative, but is an operator with a long, but fading, memory.

The second term on the left of Equation (2) represents this initialization response due to the behavior of the system before $t = 0$. It should be noticed that the past history of the particular variable that is fractionally differentiated must be known for as long as the system has been operated to obtain the correct initialization response. This equation can also be written, for $t > (c = 0)$, as

$${}_c D_t^q x(t) \equiv {}_0 d_t^q x(t) + \psi(q, x, a, 0, t) = Ax(t) + Bu(t) \quad (3)$$

where, for terminal charging,

$$\psi(q, x, a, 0, t) \equiv {}_a d_0^q x(t) = \frac{d}{dt} \left[\frac{1}{\Gamma(1-q)} \int_a^0 \frac{x(\tau)}{(t-\tau)^q} d\tau \right], \quad t > 0. \quad (4)$$

This is known as the initialization function. Note that only “terminal charging” is considered in this paper (Lorenzo and Hartley, 1998).

This initialization problem is now solved using Laplace transforms. Equation (3) can be Laplace transformed as

$$s^q X(s) + \psi(q, x, a, 0, s) = AX(s) + BU(s). \quad (5)$$

It should then be recognized from Equation (4) that the Laplace transform of the initialization function is

$$\psi(q, x, a, 0, s) = L \left\{ \frac{d}{dt} \left[\frac{1}{\Gamma(1-q)} \int_a^0 \frac{x(\tau)}{(t-\tau)^q} d\tau \right] \right\}. \quad (6)$$

Rearranging Equation (5) gives

$$X(s) = \frac{B}{s^q - A} U(s) - \frac{1}{s^q - A} \psi(q, x, a, 0, s), \quad (7)$$

which can be inverse transformed, using the Laplace transform identities of Hartley and Lorenzo (1998b), to give

$$x(t) = \int_0^t F_q[A, \tau] B u(t-\tau) d\tau - \int_0^t F_q[A, \tau] \psi(q, x, a, 0, t-\tau) d\tau. \quad (8)$$

Here the F -function is the impulse response of the fundamental linear fractional differential equation, as discussed in (Hartley and Lorenzo, 1998b). It is defined as

$$F_q[A, t] \equiv t^{q-1} \sum_{n=0}^{\infty} \frac{A^n t^{nq}}{\Gamma(nq + q)} \quad (9)$$

Clearly, the first term in Equation (8) represents any forced response due to $u(t)$, and the second term represents the initialization response of the system due to the past history of $x(t)$. Traditionally, for the integer order system ($q=1$) this initialization term becomes $\psi(1, x, a, 0, s)$ equal to a constant. Alternatively, for fractional order systems, this term is time varying into the future; that is, the past history of $x(t)$ has the appearance of a time dependent forcing term into the infinite future.

To give some additional insight into the problem, a specific function is chosen for $x(t)$, $a < t \leq 0$. Choosing the history as $x(t)=k$, a constant, for $-\infty < t \leq 0$, (with $a = -\infty$), then the initialization function of Equation (6) becomes

$$\psi(q, x, -\infty, 0, s) = L \left\{ \lim_{a \rightarrow -\infty} \frac{d}{dt} \left[\frac{1}{\Gamma(1-q)} \int_a^0 \frac{k}{(t-\tau)^q} d\tau \right] \right\}. \quad (10)$$

Performing the integration on the right side of this equation gives

$$\psi(q, x, -\infty, 0, s) = L \left\{ \lim_{a \rightarrow -\infty} \frac{d}{dt} \left[\frac{k}{\Gamma(1-q)} \frac{-(t-\tau)^{1-q}}{(1-q)} \right]_{\tau=a}^{\tau=0} \right\}. \quad (11)$$

Inserting the limits of integration gives

$$\psi(q, x, -\infty, 0, s) = L \left\{ \lim_{a \rightarrow -\infty} \frac{d}{dt} \left[\frac{k}{\Gamma(1-q)} \left(\frac{(t-a)^{1-q}}{(1-q)} - \frac{t^{1-q}}{(1-q)} \right) \right] \right\}. \quad (12)$$

Performing the required differentiation gives

$$\psi(q, x, -\infty, 0, s) = L \left\{ \lim_{a \rightarrow -\infty} \frac{k}{\Gamma(1-q)} \left(\frac{1}{(t-a)^q} - \frac{1}{t^q} \right) \right\}. \quad (13)$$

Invoking the limiting process forces the first of the two time functions to zero. Then the remaining term becomes

$$\psi(q, x, -\infty, 0, s) = -L \left\{ \frac{k}{\Gamma(1-q)} \frac{1}{t^q} \right\}. \quad (14)$$

Evaluating the Laplace transform yields the result (for $q < 1$)

$$\psi(q, x, -\infty, 0, s) = -k s^{q-1}. \quad (15)$$

Inserting this back into Equation (7) gives the Laplace transform of the total response of the system, again for the history $x(t) = k, t < 0$,

$$X(s) = \frac{B}{s^q - A} U(s) + \frac{k s^q}{s(s^q - A)}. \quad (16)$$

This equation can now be inverse Laplace transformed using the transform pairs of Hartley and Lorenzo (1998b). The total response using this specific initialization is then

$$x(t) = B \int_0^t F_q[A, \tau] u(t-\tau) d\tau + k E_q[At^q]. \quad (17)$$

Here it should be recognized that the first term is the convolution of the input $u(t)$ with the system impulse response, that is, the F -function. The second term is the initialization function and, for this particular past history, is the Mittag-Leffler E -function (Hartley and Lorenzo, 1998b).

The Vector Initialization Problem

A useful representation for systems of fractional order differential equations is the vector space of fractional dynamic variables. It should be noted that the number of fractional dynamic variables in a particular representation is somewhat arbitrary, as various sub-multiple basis values can usually be chosen for q . Once this is done, the vector representation can be written as

$${}_c D_t^q \underline{x}(t) = A \underline{x}(t) + B \underline{u}(t), \quad \underline{x}(t) \text{ given for } a \leq t \leq c, \\ \text{or } \underline{\psi}(q, \underline{x}, a, c, t) \text{ given for } t > c \quad (18)$$

$$\underline{y}(t) = C \underline{x}(t) + D \underline{u}(t) \quad (19)$$

where $\underline{x}(t)$, $\underline{y}(t)$, $\underline{u}(t)$, and $\underline{\psi}(q, \underline{x}, a, 0, t)$ can generally be considered to be vectors of the appropriate dimension, and the starting time, c , will usually be chosen as zero. Note that ${}_c D_t^q \underline{x}(t) \equiv {}_c d_t^q \underline{x}(t) + \underline{\psi}(q, \underline{x}, a, 0, t)$ where the $\underline{\psi}$ -function was defined in Equation (4). Equation (18) can be rewritten as a vector generalization of Equation (3) as

$${}_0 d_t^q \underline{x}(t) + \underline{\psi}(q, \underline{x}, a, 0, t) = A \underline{x}(t) + B \underline{u}(t), \quad (20)$$

where it is now understood that the initialization function is the vector of initialization functions,

$$\underline{\psi}(q, \underline{x}, a, 0, t) = \begin{bmatrix} \psi(q, x_1, a, 0, t) \\ \psi(q, x_2, a, 0, t) \\ \vdots \\ \psi(q, x_n, a, 0, t) \end{bmatrix} \quad (21)$$

In what follows, this will, without loss of generality, be written as

$$\underline{\psi}(q, \underline{x}, a, 0, t) = \underline{\psi}(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} \quad (22)$$

It should be noted that Bagley and Calico (1991) and Padovan and Sawicki (1997) both present a fractional state space representation, but do not include the general historic effects of the initialization function which are presented here. It should also be noticed that if the original differential equations have non-rationally related derivatives, such as ${}_0 d_t^{1/\sqrt{2}} x(t) + {}_0 d_t^{1/\pi} x(t) + x(t) = u(t)$, then this approach will require an approximation of the value of q , as integer sub-multiples of the orders are not obtainable.

At this point, it is important to notice that the fractional dynamic variables in the system of Equations (18) and (19) are not states in the true sense of the name "state" space. In the usual integer order system theory, the set of states of the system, known at any given point in time, along with the system equations, are sufficient to predict the response of the system both forward or backward in time. Otherwise stated, the collection of numbers $\underline{x}(t)$, at any time t , specify the complete "state" of the system at that time. Mathematically stated, the system will have a unique time response given its initial state. It should be clear, however, that the fractional dynamic variables do not represent the "state" of a system at any given time due to the presence of the initialization function vector, which carries information about the history of the elements of the

system. Consequently, as the initialization function vector is generally present, the set of elements of the vector $\underline{x}(t)$, evaluated at any point in time, does not specify the entire “state” of the system. Thus in a fractional system setting, the ability to predict the future response of a system requires the set of fractional differential equations along with their initialization functions, that is, Equation (18). A further implication, mentioned below, is that the impulse response of the vector system does not possess the semi-group property.

The initialization problem can be solved for $\underline{x}(t)$ as in the scalar case by using Laplace transforms, and generalizing Equation (5) for the vector initialization function,

$$s^q \underline{X}(s) + \underline{\psi}(s) = A \underline{X}(s) + B \underline{U}(s) \quad (23)$$

$$\underline{Y}(s) = C \underline{X}(s) + D \underline{U}(s). \quad (24)$$

Rearranging Equation (23) gives

$$(Is^q - A) \underline{X}(s) = B \underline{U}(s) - \underline{\psi}(s). \quad (25)$$

or equivalently

$$\underline{X}(s) = (Is^q - A)^{-1} B \underline{U}(s) - (Is^q - A)^{-1} \underline{\psi}(s). \quad (26)$$

Inserting Equation (26) into Equation (24) gives the Laplace transform of the total system output response

$$\underline{Y}(s) = \left\{ C (Is^q - A)^{-1} B + D \right\} \underline{U}(s) - C (Is^q - A)^{-1} \underline{\psi}(s). \quad (27)$$

This can now be inverse Laplace transformed to give the time response

$$\underline{y}(t) = C \int_0^t F_q[A, \tau] B \underline{u}(t - \tau) d\tau + D \underline{u}(t) - C \int_0^t F_q[A, t] \underline{\psi}(t - \tau) d\tau. \quad (28)$$

This can also be represented as

$$\underline{y}(t) = C \int_0^t F_q[A, \tau] \{ B \underline{u}(t - \tau) - \underline{\psi}(t - \tau) \} d\tau + D \underline{u}(t). \quad (29a)$$

or equivalently (using the convolution property) as

$$\underline{y}(t) = C \int_0^t F_q[A, t - \tau] \{ B \underline{u}(\tau) - \underline{\psi}(\tau) \} d\tau + D \underline{u}(t). \quad (29b)$$

Clearly this requires the use of the matrix F -function which can be obtained by the use of its series expansion. For example, the matrix F -function is defined as

$$F_q[A, t] \equiv t^{q-1} \sum_{n=0}^{\infty} \frac{A^n t^{nq}}{\Gamma(nq + n)}, \quad q > 0, \quad (30)$$

where A is now generally the square $n \times n$ system matrix. As for integer order systems, other approaches can probably be found for evaluating the matrix F – function (Moler & Van Loan, 1978), but only the Laplace transform approach will be considered further as it is easily applied (see Example below).

Example

This section presents an example to demonstrate the application of vector fractional order differential equations. The system considered is the inductor terminated lossy line studied by Heaviside (1922) and Bush (1929), and is shown in Figure 1. This system was studied in Hartley and Lorenzo (1998b), using an input-output approach. Here a vector fractional dynamic variable approach, which includes initialization effects, is presented. The vector fractional dynamic variable equations for this system are now derived.

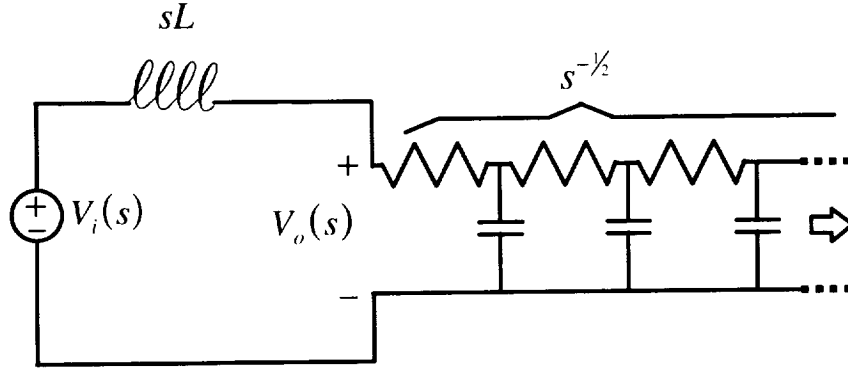


Figure 1. Inductor terminated semi-infinite lossy line example.

The dynamic relationship between current and voltage for the inductor is

$$v_L(t) = L {}_c D_t^1 i(t),$$

where L is the inductance of the coil. Likewise for the lossy line

$$v_o(t) = \alpha {}_c D_t^{-1/2} i(t),$$

where α is a line constant. In differential form, with $c = 0$, these equations can be written as

$$\begin{aligned} {}_0 d_t^1 i(t) + \psi(1, i(t), a, 0, t) &= \frac{1}{L} v_L(t) \\ {}_0 d_t^{1/2} v_o(t) + \psi(1/2, v_o(t), a, 0, t) &= \alpha i(t). \end{aligned} \quad (31)$$

However, from Kirchhoff's voltage law, $v_L(t) = v_i(t) - v_o(t)$. Replacing this in the inductor equation gives its dynamic equations as

$${}_0d_t^1 i(t) = -\frac{1}{L}v_o(t) + \frac{1}{L}v_i(t) - \psi(1, i(t), a, 0, t). \quad (32)$$

To have a vector space of fractional dynamic variables, it is necessary to reduce all of the differential relationships to differentials based on the largest common differential fraction. In this case that would be $q=1/2$. Thus we will define the following system fractional dynamic variable vector,

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \equiv \begin{bmatrix} v_o(t) \\ i(t) \\ {}_0d_t^{1/2} i(t) \end{bmatrix}. \quad (33)$$

The system input is defined as $u(t) = v_i(t)$, and the system output is chosen as $y(t) = v_o(t)$. With these definitions, Equations (31) and (32) can be written in vector representation

$${}_0d_t^{1/2} \underline{x}(t) = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 1 \\ -\frac{1}{L} & 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t) + \begin{bmatrix} -\psi(1/2, v_o(t), a, 0, t) \\ 0 \\ -\psi(1, i(t), a, 0, t) \end{bmatrix} \quad (34)$$

$$y(t) = [1 \ 0 \ 0] \underline{x}(t) + [0] u(t) \quad (35)$$

which are the specific forms of Equations (18) and (19) for this problem.

Performing the required manipulations for this problem, Equation (26) becomes

$$\underline{X}(s) = \frac{1}{s^{3/2} + \alpha/L} \begin{bmatrix} s & \alpha s^{1/2} & \alpha \\ -\frac{1}{L} & s & s^{1/2} \\ -\frac{s^{1/2}}{L} & -\frac{\alpha}{L} & s \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} U(s) - \begin{bmatrix} \psi_1(s) \\ \psi_2(s) \\ \psi_3(s) \end{bmatrix} \right\}, \quad (36)$$

where, from Eqn. (34), $\psi_1(s) = \psi(1/2, v_o(t), a, 0, s)$, $\psi_2(s) = 0$, and $\psi_3(s) = \psi(1, i(t), a, 0, s)$ which is a constant for an inductor. The Laplace transform of the forced response, $\underline{X}_F(s)$, is then the first term of Equation (36)

$$\underline{X}_F(s) = \frac{1/L}{s^{3/2} + \alpha/L} \begin{bmatrix} \alpha \\ s^{1/2} \\ s \end{bmatrix} U(s), \quad (37)$$

and the Laplace transform of the initialization response, $\underline{X}_I(s)$, is the second term of Equation (36)

$$\underline{X}_I(s) = \frac{-1}{s^{3/2} + \alpha/L} \begin{bmatrix} s\psi_1(s) + \alpha s^{1/2}\psi_2(s) + \alpha\psi_3(s) \\ \left(-\frac{1}{L}\right)\psi_1(s) + s\psi_2(s) + s^{1/2}\psi_3(s) \\ \left(-\frac{s^{1/2}}{L}\right)\psi_1(s) + \left(-\frac{\alpha}{L}\right)\psi_2(s) + s\psi_3(s) \end{bmatrix}. \quad (38)$$

These expressions can now be evaluated for any specific inputs and initialization functions.

It should be noted that the availability of fractional dynamic variables now allows many choices for the basis q , and thus the total number of fractional dynamic variables can increase if the basis q is chosen to be smaller. In the example of this section, choosing $q = 1/4$, rather than $1/2$, would yield six fractional dynamic variables rather than three. It is also important to remember that the least number of fractional dynamic variables is obtained by choosing the basis q as the largest common fraction of the differential orders. For example, given $q_1 = 1/2$, $q_2 = 1/3$, would require a basis q of at most $q = 1/6$.

Discussion

Some important observations concerning the vector space of fractional dynamic variables should be made at this point. First of all, as discussed above, it is necessary to compute matrix F -functions. The use of the infinite series is sufficient for many problems, but there are probably many other ways to perform this type of computation (Moler & Van Loan, 1978).

Questions of controllability, observability, and minimality also arise when the system is expressed in a vector fractional dynamic variable form. There are two directions to go in this regard. One is to completely rederive all of the related vector system properties. This is not done at this time. It is important to note, that the fractional dynamic variable vector alone does not contain all the information about the state of the system, but requires the addition of the ψ -vector. That is, the F -functions do not have the semi-group property (Curtain and Zwart, 1991). The implications of this are not totally clear at present; however, one important consideration in observability and controllability issues is the inclusion of the time dependent initialization function. Conversely, rather than rederiving all of the system theory results on controllability and observability, useful results can still be obtained by simply using the controllability, observability, and Hankel matrices directly for the particular given fractional vector system, while neglecting the ψ -vector. This can be done as usual, given the system (A, B, C, D) matrices,

completely without regard to their deeper system theoretic implications. In doing so, these matrices losing rank would then not necessarily imply loss of controllability or observability, but it would imply that there is a w -plane pole-zero cancellation (Hartley and Lorenzo, 1998b), as these algebraic properties of the vector space system **are** preserved. This is a very useful result! The minimality of a given system with respect to the number of basis w -plane operators can then be determined by direct consideration of the singular values of the Hankel matrix (which are probably not the same as Hankel singular values in the usual integer order system sense). It would then appear that many of the usual model reduction techniques still apply with respect to the basis w -plane operator. However, little can be said about minimality with respect to the usual integer order operators.

Finally, it should be noted that the vector space of fractional dynamic variables allows direct use of all the standard state feedback and observer theory, with the understanding that closed-loop poles are being placed in the w -plane. It is not clear at this point, however, how to interpret any of optimal control theory. Although we could use Lyapunov and Riccati equations for design, their interpretation is not at all clear with regard to optimality. One would expect that the resulting controllers, which are always guaranteed to have closed-loop poles in the left half s -plane, would now place all of the closed-loop poles in the left half of the w -plane, which would then guarantee some form of overdamped response. Clearly, the optimal control issue requires much further study.

References

- R.L. Bagley and R.A. Calico (1991), "Fractional Order State Equations for the Control of Viscoelastic Structures," *J. Guid. Cont. and Dyn.*, vol. 14, no. 2, Mar.-Apr. 1991, pp. 304-311.
- V. Bush (1929), Operational Circuit Analysis, Wiley, New York.
- H.S. Carslaw and J.C. Jeager (1948), Operational Methods in Applied Mathematics, 2nd, Oxford University Press.
- R. Courant and D. Hilbert (1953), Methods of Mathematical Physics, Interscience, New York.
- R.F. Curtain and H.J. Zwart (1991), An Introduction to Infinite-Dimensional System Theory, Springer-Verlag, New York.
- S. Goldman (1949), Transformation Calculus and Electrical Transients, Prentice-Hall, New York.
- T.T. Hartley, C.F. Lorenzo, H.K. Qammar (1995), "Chaos in a Fractional Order Chua System," *IEEE Trans. Cir. & Sys. I*, vol 42, no 8, pp 485-490, August 1995.
- T.T. Hartley and C.F. Lorenzo (1998), "Insights into the Fractional Order Initial Value Problem via Semi-infinite Systems," NASA TM-1998-208407, November 1998.
- T.T. Hartley and C.F. Lorenzo (1998b), "A Solution to the Fundamental Linear Fractional Order Differential Equation," NASA TP-1998-208693, December 1998.
- O. Heaviside (1922), Electromagnetic Theory, vol II, Chelsea Edition (1971), New York.
- J.G. Holbrook (1966), Laplace Transforms for Electronic Engineers, 2nd, Pergamon Press, New York.
- M. Ichise, Y. Nagayanagi, and T. Kojima (1971), "An Analog Simulation of Non-integer Order Transfer Functions for Analysis of Electrode Processes," *J. Electroanal. Chem. Interfacial Electrochem.*, vol 33, pp 253-265.
- R.C. Koeller (1984), "Application of Fractional Calculus to the Thoery of Viscoelasticity," *J. Appl. Mech.*, vol 51, pp 299-307, June 1984.
- R.C. Koeller (1986), "Polynomial Operators, Stieltjes Convolution, and Fractional Calculus in Hereditary Mechanics," *Acta Mechanica*, vol 58, pp 251-264.
- C.F. Lorenzo and T.T. Hartley (1998), "Initialization, Conceptualization, and Application in the Generalized (Fractional) Calculus," NASA TP-1998-208415, December 1998.
- B. Mandelbrot (1967), "Some Noises with 1/f Spectrum, a Bridge Between Direct Current and White Noise," *IEEE Trans. Inf. Theory*, vol 13, no 2, pp 289-298, April 1967.
- J. Mikusinski (1959), Operational Calculus, Pergamon Press, New York.
- K.S. Miller and B. Ross (1993), An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York.

C.B. Moler and C.F. Van Loan (1978), "Nineteen Dubious Ways to Compute the Exponential of a Matrix," SIAM Rev. vol 20, pp 801-836, Oct. 1978.

K. B. Oldham and J. Spanier (1974), The Fractional Calculus, Academic Press, San Diego.

J. Padovan and J.T. Sawicki (1997), "Diophantine type fractional derivative representation of structural dynamics, Part I: Formulation" Computational Mechanics, vol 19, pp 335-340.

E.J. Scott (1955), Transform Calculus with an Introduction to Complex Variables, Harper, New York.

S.B. Skaar, A.N. Michel, and R.K. Miller, "Stability of Viscoelastic Control Systems," IEEE Trans. Auto. Cont., vol 33, no 4, pp 348-357, April 1988.

B.J. Starkey, Laplace Transforms for Electrical Engineers, Iliffe, London.

H.H. Sun, A.A. Abdelwahab, and B. Onaral, "Linear Approximation of Transfer Function with a Pole of Fractional Order," IEEE Trans. Auto. Cont., vol 29, no 5, pp 441-444, May 1984.

H.H. Sun, B. Onaral, and Y. Tsao, "Application of Positive Reality Principle to Metal Electrode Linear Polarization Phenomena," IEEE Trans. Biomed. Eng., vol 31, no 10, pp 664-674, October, 1984.

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE May 1999		3. REPORT TYPE AND DATES COVERED Technical Paper
4. TITLE AND SUBTITLE The Vertical Linear Fractional Initialization Problem			5. FUNDING NUMBERS WU-523-22-13-00	
6. AUTHOR(S) Tom T. Hartley and Carl F. Lorenzo				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration John H. Glenn Research Center at Lewis Field Cleveland, Ohio 44135-3191			8. PERFORMING ORGANIZATION REPORT NUMBER E-11699	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001			10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA TP-1999-208919	
11. SUPPLEMENTARY NOTES Tom T. Hartley, Department of Electrical Engineering, The University of Akron, Akron, Ohio 44325-3904; and Carl F. Lorenzo, NASA Glenn Research Center. Responsible person, Carl F. Lorenzo, organization code 5500, (216) 433-3733.				
12a. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified - Unlimited Subject Categories: 59, 66, 31, and 70 This publication is available from the NASA Center for AeroSpace Information, (301) 621-0390.			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) This paper presents a solution to the initialization problem for a system of linear fractional-order differential equations. The scalar problem is considered first, and solutions are obtained both generally and for a specific initialization. Next the vector fractional order differential equation is considered. In this case, the solution is obtained in the form of matrix F-functions. Some control implications of the vector case are discussed. The suggested method of problem solution is shown via an example.				
14. SUBJECT TERMS Fractional calculus; Fractional differential equations; Systems; Eigenfunction			15. NUMBER OF PAGES 18	
			16. PRICE CODE A03	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT	